Perfect(oid)ness, . . .

<u>Recall</u> On $\operatorname{Spec} A$ we have a basis of open subsets $\operatorname{Spec} A_f$ w/ A_f the "functions" on $\operatorname{Spec} A_f$.

On $\operatorname{Spa}(A, A^+)$ we have open subsets

$$\operatorname{Spa}\left(A\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle, A\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle^+ \right)$$

where $A \longrightarrow A\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle$ is a "rational localization": $(f_1, \dots, f_n)A = A$.

A is <u>stably uniform</u> if $A\langle f_\bullet / g \rangle$ is uniform for any $(f_1, \dots, f_n)A = A$ and $g \in A$.

<u>Thm</u> if A is stably uniform then the structure presheaf on $\operatorname{Spa}(A, A^+)$ is a sheaf.

Def ring R of char p is *perfect*

if $\text{Frob} = \bar{\varphi} : R \to R$ is an isom.

[inj. is equivalent to R reduced].

We have "perfection" operations:

$$R^{perf} = \varinjlim_{\bar{\varphi}} R. \quad [R \text{ char } p]$$

$$R^{frep} = \varprojlim_{\bar{\varphi}} R/(p) \quad [R \text{ arbitrary}].$$

If R char. p

we have $R \to R^{perf}$ and $R^{frep} \to R$

and these are initial / final

among maps to / from perfect rings.

Note: any power-mult seminorm on R extends

uniquely to R^{perf} and get

$\mathcal{M}(R) \cong \mathcal{M}(R^{perf})$ [homeomorphism]

<u>Lemma</u> if R perfect [char p] and

$R \to S$ is

- finite etale, or
- rational localization

then S is also perfect

<u>Def</u> A strict p-ring is a p-torsion-free, p-adically complete ring A s.t. A/p perfect.

<u>Lemma</u> if $\bar{t} : A/p \to B/p$ w/ A strict p-ring and B p-adically complete, then $\exists!$ multiplicative lift $t : A/p \to B$.

Pf if $\bar{x} \in A/p$ then let

y_n = lift of $\bar{t}(\bar{x}^{1/p^n})$ to B,

then $y_n^{p^n} \equiv \bar{t}(\bar{x}^{1/p^n})^{p^n} \mod p^{n+1}B$

$\equiv \bar{t}(\bar{x}) \mod p^{n+1}B.$

Therefore $\{y_n^{p^n}\} \longrightarrow y$ w/ $y \equiv \bar{t}(\bar{x}) \mod p.$

Now if A strict p-ring and take

$A/p \xrightarrow{id} A/p$ we get induced

[mult.] map $A/p \xrightarrow{[\cdot]} A$, $\overline{[x]} = x.$

Rmk if $a \in A$ then it has a unique

expansion $a = \sum_{n=0}^{\infty} [\bar{a}_n] p^n.$

$\bar{a}_0 = \bar{a}$. Then $a \equiv [\bar{a}_0] \mod p$

So $a = [\bar{a}_0] + p a_1 \quad a_1 \in A$.

$a_1 \equiv [\bar{a}_1] \mod p \rightsquigarrow a_1 = [\bar{a}_1] + p a_2$.

$$
\begin{aligned}
\Rightarrow a &= [\bar{a}_0] + p([\bar{a}_1] + p a_2) \\
&= [\bar{a}_0] + p[\bar{a}_1] + p^2 a_2 + \dots
\end{aligned}
$$

Now if $\bar{t} : A/p \to B/p$ then

$$
\begin{tikzcd}
A \arrow[r, "T := t \circ \pi_A"] \arrow[d, "\pi_A"'] & B \arrow[d] \\
A/p \arrow[r, "\bar{t}"'] \arrow[ur, dashed, "t"] & B/p
\end{tikzcd}
$$

Explicitly

$$T\Big(\sum_n [\bar{a}_n] p^n\Big) = \sum_n [\bar{t}(\bar{a}_n)] p^n.$$

<u>Thm</u> There is an equivalence

$$
\begin{aligned}
(\text{strict } p\text{-rings}) &\longrightarrow (\text{perfect } \mathbb{F}_p\text{-alg}) \\
A &\longmapsto A/p.
\end{aligned}
$$

We just proved fully faithfulness.

If R perfect $\mathbb{F}_p$-alg we write

$W(R)$ for corresponding strict p-ring

w/ $W(R)/p \cong R$.

<u>Def</u> For A ring let $\hat{A}$ be p-adic completion

From $A^{frep} = \varprojlim_{\varphi} A/p \twoheadrightarrow A/p = \hat{A}/p$.

then get $\theta : W(A^{frep}) \to \hat{A}$.

$$\begin{tikzcd}
W(A^{frep}) \arrow[r, dashed, "\theta"] \arrow[d] & \hat{A} \arrow[d] \\
A^{frep} \arrow[ur, dashed, "t"] \arrow[r, "\bar{t}"] & \hat{A}/p
\end{tikzcd}$$

Fact θ is surj. $\Leftrightarrow$ $\bar{\varphi} : A/p \to A/p$ is surj.

$\longrightarrow$ X $\longrightarrow$

Have maps

$$\mathcal{M}(R) \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \mathcal{M}(W(R))$$

$\alpha \in \mathcal{M}(R)$, $\beta \in \mathcal{M}(W(R))$

$\mu \circ \lambda = \mathrm{id}$ $\qquad$ $\lambda \circ \mu \geq \mathrm{id}$

$$\lambda(\alpha)\left(\sum_n [\bar{x}_n] p^n\right) = \sup_n \{ p^{-n} \alpha(\bar{x}_n) \}.$$

$$\mu(\beta)(x) = \beta([x]).$$

if α is [blah] (semi)norm so is $\lambda(\alpha)$.

And similarly for $\beta \rightsquigarrow \mu(\beta)$.

if A p-adically separated β is [blah] (semi)norm on A bdd by p-adic norm, extend β to $\hat{A}$. Then $\theta^*(\beta)$ on $W(A^{\mathrm{perf}})$ and $\alpha = \mu(\theta^*\beta)$ is [blah] seminorm on A^{perf}.

Def if R perfect $/\mathbb{F}_p$ and complete

wrt power-mult norm α bdd by

trivial norm then say

$z = \sum_n [\bar{z}_n] p^n \in W(R)$ is

<u>primitive of deg 1</u> if

$\alpha(\overline{x z_0}) = p^{-1} \alpha(\bar{x}) \quad \forall x \in R$

and $\bar{z}_1 \in R^\times$.

Prop if A p-adically separated, p-torsion free,

complete for power-mult β letting

$\alpha = \mu(\theta^* \beta)$, suppose $\exists z \in W(A^{\text{frep}})$

prim. deg 1 and $z \in \ker(\theta : W(A^{\text{frep}}) \to \hat{A})$.

Then z generates the kernel of

$\theta : W(A^{\text{frep}})[[\bar{z}]^{-1}] \to \hat{A}[\theta([\bar{z}]^{-1})]$

Def A perfectoid field is an analytic field F of residue char p s.t.

F <u>not</u> discretely valued, $\bar{\varphi}$ surj. on $\mathcal{O}_F/p$.

[$\Rightarrow$ equiv. $\mathcal{O}_F$ not Noeth]

Rmk in char p this just means perfect.

For θ we defined earlier w/ $A = \mathcal{O}_F$

get $\theta : W(\mathcal{O}_F^{\text{frep}}) \to \hat{\mathcal{O}}_F = \mathcal{O}_F$.

$\bar{\varphi} : \mathcal{O}_F/p \to \mathcal{O}_F/p$ surj. $\Rightarrow$ θ surj.

if β norm on F get $\alpha = \mu(\theta^*\beta)$

norm on $\mathcal{O}_F^{\text{frep}} = \varprojlim_{\bar{\varphi}} \mathcal{O}_F/p$

So $F^\flat := \text{Frac}(\mathcal{O}_F^{\text{frep}})$ is an analytic perfect (ed) field in char p. $\mathcal{O}_F^{\text{frep}} = \mathcal{O}_{F^\flat}$,

Lemma i) $F^\flat$ is perfectoid

ii) $\beta(F) = \alpha(F^\flat)$

iii) for $z \in F^\flat$ w/ $\alpha(z) = p^{-1}$

get $\mathcal{O}_F/p \xrightarrow{\sim} \mathcal{O}_{F^\flat}/z$

iv) $\exists\, z \in W(\mathcal{O}_{F^\flat})$ prim of deg 1

which is in $\ker\theta$.

Thus $\ker(\theta : W(\mathcal{O}_{F^\flat})[[z]^{-1}] \to F)$

is generated by prim. elt deg 1.

Thm We have an equivalence

$$\begin{pmatrix} \text{perfectoid in} \\ \text{char } 0 \\ \text{residue } p \end{pmatrix} \longrightarrow \begin{pmatrix} \text{pairs } (L, I) \\ \text{w/ } L \text{ perfectoid} \\ \text{char } p \text{ and } I \\ \text{ideal of } W(\mathcal{O}_L) \text{ gen.} \\ \text{by prim. elt. deg } 1 \end{pmatrix}$$

$F \mapsto (F^\flat, \ker\theta)$

$(L, I) \mapsto \mathrm{Frac}(W(\mathcal{O}_L)/I)$

$\natural \quad F \longleftrightarrow (L, \mathcal{E})$

<u>Thm</u> $FEt(F) \cong FEt(L)$.

<u>Cor</u> $Gal_F \simeq Gal_L$.

<u>Thm</u> if E/F finite ext. of analytic fields, E perfd $\Leftrightarrow$ F is.

<u>Def</u> A (Tate) Banach pair (A, A^+) perfd if uniform & $\exists\ \varpi$ pseudo-unif. s.t. $\varpi^p \mid p$ in A^+ and

$$A^+/\varpi \xrightarrow{\bar{\varphi}} A^+/\varpi^p \text{ is surj.}$$

<u>Rmk</u> ind. of A^+.

<u>Rmk</u> in char p again $\Longleftrightarrow$ A perfect

Again we get

$z \in \ker(\theta : W(A^{+\flat}) \to A^{+\flat})$

prim. deg 1 $\Rightarrow$ z generates

$\ker\theta \subseteq W(A^{+\flat})[[z]^{-1}]$

Let $A^{\flat +} = A^{+\flat}$ $\quad A^{\flat} = A^{+\flat}[z^{-1}]$.

Then (A, A^{+}) perf'd char p

<u>Thm</u> $(A, A^{+}) \longmapsto ((A^{\flat}, A^{\flat +}), \ker\theta)$

equivalence of

$$\left(\begin{array}{c}\text{perfectoid pairs} \\ \text{char } 0\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{pairs } ((A, A^{+}), I) \text{ w/} \\ \text{perf'd pairs and} \\ I \subset W(A^{+}) \text{ principal} \\ \text{gen. by prim. dg } 1.\end{array}\right)$$

<u>Thm</u> $FEt(A) \xrightarrow{\sim} FEt(A^{\flat})$

<u>Thm</u> A perf'd $\Rightarrow$ $A\langle f_0/s\rangle$ perf'd.

Get A stably uniform $\Rightarrow$ sheafy.

Thm if $A \to B$ finite etale

A perfd $\Rightarrow$ B perfd.

Thm $\mathcal{M}(A) \overset{\text{homeomorphism}}{\cong} \mathcal{M}(A^\flat)$

$\mathrm{Spa}(A, A^+] \cong \mathrm{Spa}(A^\flat, A^{\flat +})$.